University of Insubria

DEPARTMENT OF SCIENCE AND HIGH TECHNOLOGY
PhD in Computer Science and Mathematics of Calculus

Exam of the RISM Course III

Generalized Solutions in Differential Equations: Theory and Applications

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Exercise 1. Prove that any Non-Archimedean field cannot satisfy the Dedekind's axiom.

Solution. Let be $\mathbb{K}$ a Non-Archimedean field. So $\mathbb{K} \equiv (\mathbb{K}, +, \cdot, \leq)$ is an infinite totally ordered field such that there exists at least an infinitesimal number $\xi \in \mathbb{K} \setminus \{0\}$. We can suppose $\xi > 0$ and we know that, for every $N \in \mathbb{N} \setminus \{0\}$, it holds $\xi < \frac{1}{N}$ (where: $0 \in \mathbb{K}$ denotes the neutral element with respect to $+$, $1 \in \mathbb{K}$ denotes the neutral element with respect to $\cdot$, $N = 1 + \cdots + 1$ (N times) and $\frac{1}{N} = N^{-1}$ w.r.t. $\cdot$).

Now let's consider the two following countable subsets of $\mathbb{K}$:

$$A_\xi \doteq \left\{ -\frac{\xi}{n} \;\middle|\; n \in \mathbb{N} \setminus \{0\} \right\} \quad \text{and} \quad B_\xi \doteq \left\{ -m\,\xi^2 \;\middle|\; m \in \mathbb{N} \right\}.$$

Then it's easy to verify that B_ξ is a set of majorant elements for A_ξ in the sense that ($-\xi < 0$ and), for every $n, m \in \mathbb{N} \setminus \{0\}$, $-\frac{\xi}{n} < -m\,\xi^2$ (because equivalently $\xi < \frac{1}{nm}$).

Finally, let be $x \in \mathbb{K}$, $x > 0$, another generic element among those for which $-x$ is a majorant element for A_ξ: this means that, for every $n \in \mathbb{N} \setminus \{0\}$, $-\frac{\xi}{n} < -x$ (in particular, x is an infinitesimal number). Then, for every $m \in \mathbb{N} \setminus \{0\}$, $-x - m\,\xi^2$ is a majorant element for A_ξ as well: indeed, for every $n \in \mathbb{N} \setminus \{0\}$,

$$-\frac{\xi}{n} = \left(-\frac{\xi}{2n}\right) + \left(-\frac{\xi}{2n}\right) < -x - m\,\xi^2.$$

This shows that $-x$ can't be the minimum of all the majorant elements for A_ξ (because $-x - m\,\xi^2 < -x$) and that, consequently, $\mathbb{K}$ cannot satisfy the Dedekind's axiom. □

Exercise 2. Find an ultrafunction $u(x)$ such that

$$\oint_\Gamma |\mathrm{D}u(x)|^2\,dx = 1$$

and such that

$$\forall x \in \Gamma,\ u(x) \sim 0.$$

Solution. Let be Λ an infinite set with $\Lambda \supsetneq \mathbb{R}$ and let denote $\mathcal{L} \equiv \mathcal{L}_\Lambda \coloneqq \mathcal{P}_{\mathrm{fin}}(\Lambda)$ in such a way that, for every $\lambda \in \mathcal{L}$, λ is a finite subset of Λ. Note that $\mathcal{L}$ is a directed upward set with respect to $\subseteq$ (in fact, we could write n instead of λ thinking about the correspondence $n = |\lambda| \equiv \#\lambda \in \mathbb{N}$).

Thanks to the rings basic theory on maximal ideals, and in particular to the Krull-Zorn lemma, we know that there exists a field $\mathbb{E} \equiv \mathbb{E}_\Lambda \supsetneq \mathbb{R}$ of Euclidean numbers: that is, a totally ordered field $\mathbb{E} \equiv (\mathbb{E}, +, \cdot, \leq)$ such that can be built a surjective homomorphism $\mathrm{J} \colon \mathbb{R}^{\mathcal{L}} \to \mathbb{E}$ (between ordered algebras).

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So, given a net $\varphi\colon \mathcal{L} \to \mathbb{R}$, the Λ-limit of φ is by definition

$$\lim_{\lambda\uparrow\Lambda} \varphi(\lambda) \doteq \mathrm{J}(\varphi) \in \mathbb{E}.$$

Therefore, in fact, the Λ-limit of nets (always exists and) satisfies all the "basic properties" of an "usual limit".

Remember also that, for every Euclidean number $\xi \in \mathbb{E}$ which is finite (not infinite), there exists one and only one real number $\mathrm{st}(\xi) \in \mathbb{R}$ such that $\xi \sim \mathrm{st}(\xi)$ in the sense of infinitely closeness: this means that $\xi - \mathrm{st}(\xi)$ is an infinitesimal number of $\mathbb{E}$ (in particular, $\mathbb{E}$ is in any case a Non-Archimedean field).

As a remarkable situation of this, if for every net $\varphi\colon \mathcal{L} \to \mathbb{R}$ we denote $\lim_{\lambda\to\Lambda} \varphi(\lambda)$ the usual Cauchy-limit of φ when it exists (so, when it belongs to $\mathbb{R}$, $\forall\, \varepsilon > 0, \exists\, \lambda_\varepsilon \in \mathcal{L} : \forall\, \lambda' \in \mathcal{L},\, \lambda' \supseteq \lambda_\varepsilon \Rightarrow |\varphi(\lambda') - \lim_{\lambda\to\Lambda} \varphi(\lambda)| \leq \varepsilon$), and if $\lim_{\lambda\uparrow\Lambda} \varphi(\lambda)$ is a finite Euclidean number, then $\lim_{\lambda\to\Lambda} \varphi(\lambda) \in \mathbb{R}$ and

$$\mathrm{st}\left(\lim_{\lambda\uparrow\Lambda} \varphi(\lambda)\right) = \lim_{\lambda\to\Lambda} \varphi(\lambda).$$

Let's consider now the interval $\Omega \coloneqq \,]0,1[\, \subset \mathbb{R}$ and its natural extension Ω^* in $\mathbb{E}$, that means the set

$$\Omega^* \doteq \mathrm{J}(\Omega^{\mathcal{L}}) = \left\{ \lim_{\lambda\uparrow\Lambda} \varphi(\lambda) \;\middle|\; \varphi \in \Omega^{\mathcal{L}} \right\} \subset \mathbb{E},$$

and let be Γ a hyperfinite grid on Ω: that is, $\Omega \subset \Gamma \subset \Omega^*$ and there exists a family $\{\, \Gamma_\lambda \,\}_{\lambda\in\mathcal{L}}$ of finite subsets of Ω, so $\Gamma_\lambda \subset \Omega$ with $|\Gamma_\lambda| < \infty$ for every $\lambda \in \mathcal{L}$, such for which

$$\Gamma = \lim_{\lambda\uparrow\Lambda} \Gamma_\lambda \doteq \left\{ \lim_{\lambda\uparrow\Lambda} x_\lambda \;\middle|\; \forall\, \lambda \in \mathcal{L},\ x_\lambda \in \Gamma_\lambda \right\}.$$

For instance, we could imagine $\Gamma \equiv \Gamma_\Omega \doteq \lim_{\lambda\uparrow\Lambda}(\Omega \cap \lambda)$.

So let call grid function on Γ any function $u\colon \Omega^* \to \mathbb{E}$ such that there exists a family $\{\, u_\lambda \,\}_{\lambda\in\mathcal{L}}$ of functions $u_\lambda\colon \Omega \to \mathbb{R}$, $\lambda \in \mathcal{L}$, such for which

$$u\big|_\Gamma = \lim_{\lambda\uparrow\Lambda} u_\lambda$$

in the sense that, for every $x = \lim_{\lambda\uparrow\Lambda} x_\lambda \in \Gamma$ (where $x_\lambda \in \Gamma_\lambda$ for every $\lambda \in \mathcal{L}$), it holds

$$u(x) = \lim_{\lambda\uparrow\Lambda} u_\lambda(x_\lambda).$$

Note that, for every function $f\colon \Omega \to \mathbb{R}$, its natural extension $f^*\colon \Omega^* \to \mathbb{E}$ on Ω^* given by

$$f^*\left(\lim_{\lambda\uparrow\Lambda} x_\lambda\right) \doteq \lim_{\lambda\uparrow\Lambda} f(x_\lambda), \quad x_\lambda \in \Omega,$$

is trivially a grid function on Γ. About that, let's observe that in fact it would be sufficient to define a grid function u only on Γ, starting from $\{\, u_\lambda \,\}_{\lambda\in\mathcal{L}}$ and defining then u on Γ by $u \doteq \lim_{\lambda\uparrow\Lambda} u_\lambda$, to finally take $\left(u\big|_\Omega\right)^*$ as a function defined on the whole set Ω^*.

Now let's consider the algebra $V^\circ(\Omega)$ of ultrafunctions on Γ modelled on the pair $(V(\Omega), \{\, V_\lambda(\Omega) \,\}_{\lambda\in\mathcal{L}})$ where $V(\Omega) \coloneqq C^1_c(\Omega;\mathbb{R})$ and $\{\, V_\lambda(\Omega) \,\}_{\lambda\in\mathcal{L}}$ is a directed upward family of finite-dimensional subspaces $V_\lambda(\Omega)$ of $V(\Omega)$ which contains $\mathrm{Span}_{\mathbb{R}}(V(\Omega) \cap \lambda)$, $\lambda \in \mathcal{L}$, and with $\bigcup_{\lambda\in\mathcal{L}} V_\lambda(\Omega) = V(\Omega)$: so $u \in V^\circ(\Omega)$ if and only if u is a grid function on Γ such that, more precisely, there exists a family $\{\, u_\lambda \,\}_{\lambda\in\mathcal{L}}$ of functions

$$u_\lambda \in V_\lambda(\Omega), \quad \lambda \in \mathcal{L},$$

such for which $u\big|_\Gamma = \lim_{\lambda\uparrow\Lambda} u_\lambda$.

Finally let's choose an ultrafunction $d \in V^\circ(\Omega)$ with $d\big|_\Gamma = \lim_{\lambda\uparrow\Lambda} d_\lambda$ ($d_\lambda \in V_\lambda(\Omega)$). Then, for every $u \in V^\circ(\Omega)$ with $u\big|_\Gamma = \lim_{\lambda\uparrow\Lambda} u_\lambda$ ($u_\lambda \in V_\lambda(\Omega)$), the generalized integral on Γ of any u_λ is given by

$$\oint_\Gamma u_\lambda(x)\, dx \doteq \lim_{\lambda'\uparrow\Lambda} \sum_{a_{\lambda'}\in\Gamma_{\lambda'}} u_\lambda(a_{\lambda'}) d_\lambda(a_{\lambda'}) \in \mathbb{E},$$

and thus, when $\oint_\Gamma u_\lambda(x)\,dx \in \mathbb{R}$ for every $\lambda \in \mathcal{L}$, the generalized integral on Γ of u is given by

$$\oint_\Gamma u(x)\,dx \doteq \lim_{\lambda\uparrow\Lambda} \oint_\Gamma u_\lambda(x)\,dx \equiv \sum_{a\in\Gamma} u(a)d(a) \in \mathbb{E}$$

(the last equality follows by definition of hyperfinite sum on Γ). In particular, for every $a \in \Gamma$, it holds $d(a) = \oint_\Gamma \mathbb{1}_a(x)\,dx$ where $\mathbb{1}_a \equiv \mathbb{1}_{\{\,a\,\}}\colon \Omega^* \to \{\,0,1\,\}$ is the usual indicator function of $\{\,a\,\} \subset \Omega^*$.

Actually, we shall consider $\Gamma = \Gamma_\Omega$ and $\{\,V_\lambda(\Omega)\,\}_{\lambda\in\mathcal{L}}$, $d \in V^\circ(\Omega)$ such for which this generalized integral extends the usual Cauchy-Riemann integral; consequently, $\oint_\Gamma u_\lambda(x)\,dx = \int_\Omega u_\lambda(x)\,dx \in \mathbb{R}$ for every $\lambda \in \mathcal{L}$.

Moreover let's recall the generalized derivative $\mathrm{D}u$ as the grid function on Γ determined by

$$\mathrm{D}u\big|_\Gamma \doteq \lim_{\lambda\uparrow\Lambda} \mathrm{D}u_\lambda.$$

Observe that, if we let $|\cdot| \coloneqq (|\cdot|)^*\colon \mathbb{E} \to \mathbb{E}$ denote the natural extension on $\mathbb{R}^* \equiv \mathbb{E}$ of the usual norm $|\cdot|\colon \mathbb{R} \to [0,+\infty[$, and similarly about the second power function on $\mathbb{R}$, then by definitions

$$|\mathrm{D}u|^2\big|_\Gamma = \lim_{\lambda\uparrow\Lambda} |\mathrm{D}u_\lambda|^2$$

and in particular $|\mathrm{D}u|^2$ is a grid function on Γ as well.

Finally, keeping in mind the well known Lavrentiev phenomenon (developed with hyperfinite analysis), let's take two sequences $\{\,V_\lambda(\Omega)\,\}_{\lambda\in\mathcal{L}}$, $\{\,u_\lambda\,\}_{\lambda\in\mathcal{L}}$ which satisfy the three following properties:

- for every $\lambda \in \mathcal{L}$, u_λ and $|\mathrm{D}u_\lambda|^2$ belong to $V_\lambda(\Omega)$ and u_λ has compact support that is independent from λ;
- $\lim_{\lambda\to\Lambda} u_\lambda = 0$ uniformly;
- $\lim_{\lambda\to\Lambda} \int_\Omega |\mathrm{D}u_\lambda|^2(x)\,dx = 1$.

Then the ultrafunction $u \in V^\circ(\Omega)$ defined by $u\big|_\Gamma \overset{def}{=} \lim_{\lambda\uparrow\Lambda} u_\lambda$ demonstrates the desired statement. In fact, on the one hand, $|\mathrm{D}u|^2 \in V^\circ(\Omega)$ too and, by definitions and assumptions,

$$\oint_\Gamma |\mathrm{D}u|^2(x)\,dx = \lim_{\lambda\uparrow\Lambda} \oint_\Gamma |\mathrm{D}u_\lambda|^2(x)\,dx = 1;$$

on the other hand, for every $x = \lim_{\lambda\uparrow\Lambda} x_\lambda \in \Gamma$ (where $x_\lambda \in \Gamma_\lambda$ for every $\lambda \in \mathcal{L}$), $u(x) \equiv \lim_{\lambda\uparrow\Lambda} u_\lambda(x_\lambda)$ is a finite Euclidean number (because $\{\,u_\lambda\,\}_{\lambda\in\mathcal{L}}$ is uniformly bounded on Ω) and it has

$$\mathrm{st}\,\big(u(x)\big) \equiv \mathrm{st}\,\Big(\lim_{\lambda\uparrow\Lambda} u_\lambda(x_\lambda)\Big) = \lim_{\lambda\to\Lambda} u_\lambda(x_\lambda) = 0$$

(about the last equality, use that x is finite with $\mathrm{st}(x) = \lim_{\lambda\to\Lambda} x_\lambda \in \Omega \cup \{\,0,1\,\}$ and so, even if $\mathrm{st}(x) \notin \{\,0,1\,\}$, $|u_\lambda(x_\lambda)| \le |u_\lambda(x_\lambda) - u_\lambda(\mathrm{st}(x))| + |u_\lambda(\mathrm{st}(x))| = \mathcal{O}(|x_\lambda - \mathrm{st}(x)|) + |u_\lambda(\mathrm{st}(x))| \to 0$ for $\lambda \to \Lambda$). □

Exercise 3. Study by means of ultrafunctions a PDE which doesn't have solutions in any distributions space.

Solution. Let be $N \in \mathbb{N} \setminus \{\,0\,\}$, $\Omega \subset \mathbb{R}^N_{\boldsymbol{x}}$ a bounded open set, $u_0 \in V^\circ_0(\Omega)^L$, $T \in\,]0,+\infty[$, $I \coloneqq [0,T]_t$ and $a\colon \mathbb{R}_u \to \mathbb{R}$ the quadratic polynomial function defined by, for every $u \in \mathbb{R}$,

$$a(u) = u^2 - u$$

and thus let's consider the associated evolution problem

$$\begin{cases} u \in C^1\big(I^*, V^\circ_0(\Omega)^L\big) \\ \partial^*_t u = \mathrm{D}_{\boldsymbol{x}} \cdot \big[a^*(u)\,\mathrm{D}_{\boldsymbol{x}} u\big] \text{ on } I^* \times (\Gamma \cap \Omega^*) \\ u(0,\cdot) \equiv u_0(\cdot) \text{ on } \Gamma \cap \Omega^*. \end{cases}$$

It could be shown that there exists an unique global in time ultrafunction solution u such for which, on I^*,

$$\partial^*_t \oint_\Gamma u^2(\boldsymbol{x})\,d\boldsymbol{x} = \mathcal{O}(1).$$

Proposition. *Let's assume that* $0 < u < 1$ *on* $I^* \times (\Gamma \cap \Omega^*)$ *and that, on* I^*,

$$\partial_t^* \oint_\Gamma u^2(\boldsymbol{x})\, d\boldsymbol{x} \equiv 0 \quad \textit{and} \quad \oint_\Gamma |\mathrm{D}_{\boldsymbol{x}} u|^2(\boldsymbol{x})\, d\boldsymbol{x} = \mathcal{O}(1).$$

Then $\mathrm{D}_{\boldsymbol{x}} u \equiv 0$ *on* $I^* \times (\Gamma \cap \Omega^*)$.

In fact, let's consider the quartic polynomial function $P\colon \mathbb{R}_u \to \mathbb{R}$ given by, for every $u \in \mathbb{R}$,

$$P(u) = \frac{u^3}{6} - \frac{u^4}{12} - \frac{u^2}{2}$$

in such a way that $P''(\,\cdot\,) \equiv -\big[a(\,\cdot\,) + 1\big]$ and therefore, on I^*,

$$\partial_t^* \oint_\Gamma P(u)\, d\boldsymbol{x} = \oint_\Gamma d_u^* P(u)\, \partial_t^* u\, d\boldsymbol{x} = -\oint_\Gamma (d_u^*)^2 P(u)\, a^*(u) |\mathrm{D}_{\boldsymbol{x}} u|^2 d\boldsymbol{x} \equiv \oint_\Gamma \big[a^*(u) + 1\big] a^*(u) |\mathrm{D}_{\boldsymbol{x}} u|^2 d\boldsymbol{x}$$

while also

$$\partial_t^* \oint_\Gamma P(u)\, d\boldsymbol{x} \equiv \partial_t^* \oint_\Gamma \left(\frac{u^3}{6} - \frac{u^4}{12}\right) d\boldsymbol{x} = \oint_\Gamma (a^*)^2(u) |\mathrm{D}_{\boldsymbol{x}} u|^2 d\boldsymbol{x}$$

and ultimately

$$\oint_\Gamma a^*(u) |\mathrm{D}_{\boldsymbol{x}} u|^2 d\boldsymbol{x} \equiv 0$$

which is possible if and only if $|\mathrm{D}_{\boldsymbol{x}} u|^2 \equiv 0$ on $I^* \times (\Gamma \cap \Omega^*)$, and this concludes. □

References

[1] V. Benci. *An improved setting for generalized functions: robust ultrafunctions.* Preliminary draft, 2019.